

Mathematics: analysis and approaches**Standard Level****Paper 2 (set C)****worked solutions**

1. [Maximum mark: 6]

$$\text{Let } f(x) = \frac{x+8}{x} \text{ and } g(x) = 1-x^2.$$

$$(a) \text{ Show that } f^{-1}(x) = \frac{8}{x-1}. \quad [3]$$

$$(b) \text{ (i) Write down } (f^{-1} \circ g)(x).$$

$$(ii) \text{ Solve the equation } (f^{-1} \circ g)(x) = x. \quad [3]$$

Solution:

$$(a) \ y = \frac{x+8}{x} \Rightarrow x = \frac{y+8}{y} \Rightarrow xy = y+8 \Rightarrow xy - y = 8 \Rightarrow y(x-1) = 8$$

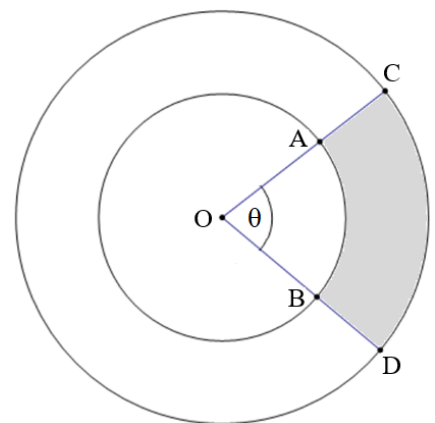
$$\text{Thus, } f^{-1}(x) = \frac{8}{x-1} \quad \text{QED}$$

$$(b) \text{ (i) } (f^{-1} \circ g)(x) = \frac{8}{(1-x^2)-1} = -\frac{8}{x^2}$$

$$(ii) \ -\frac{8}{x^2} = x \Rightarrow x^3 = -8 \Rightarrow x = -2$$

2. [Maximum mark: 6]

The diagram below shows two circles which have the same centre O. The smaller circle has a radius of 12 cm and the larger circle has a radius of 20 cm. The two arcs AB and CD have the same central angle θ , where $\theta = 1.3$ radians. Find the area of the shaded region.

**Solution:**

area of shaded region = (area of sector COD) – (area of sector AOB)

$$\text{area of sector COD} = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 20^2 \cdot 1.3 = 260 \text{ cm}^2$$

$$\text{area of sector AOB} = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 12^2 \cdot 1.3 = 93.6 \text{ cm}^2$$

$$\text{thus, area of shaded region} = 260 - 93.6 = 166.4 \approx 166 \text{ cm}^2$$

3. [Maximum mark: 5]

Find the coefficient of the x^3 term in the expansion of $\left(\frac{2}{3}x + 3\right)^8$.

Solution:

general term of expansion: $\binom{8}{r} \left(\frac{2}{3}x\right)^{8-r} 3^r$

considering exponent of x : $8 - r = 3 \Rightarrow r = 5$

thus, coefficient of x^3 term = $\binom{8}{5} \left(\frac{2}{3}\right)^3 3^5 = 56 \cdot \frac{8}{27} \cdot 243 = 4032$

4. [Maximum mark: 6]

A car begins moving from a fixed point A. Its velocity, $v \text{ ms}^{-1}$, after t seconds is given by $v = 8 - t^2 - 8e^{-t}$.

(a) Find the car's displacement from A when $t = 4$. [3]

(b) Find the total distance that the car has travelled from A when $t = 4$. [3]

Solution:

(a) displacement = $\int_0^4 (8 - t^2 - 8e^{-t}) dt \approx 2.81319\dots$

displacement from A when $t = 4$ is approximately 2.81 meters

(b) total distance = $\int_0^4 |8 - t^2 - 8e^{-t}| dt \approx 12.34617\dots$

total distance from A when $t = 4$ is approximately 12.3 meters

5. [Maximum mark: 6]

The table below shows the marks earned on a quiz by a group of students.

Mark	1	2	3	4	5
Number of students	8	7	c	9	1

The median is 3 and the mode is 4 for the set of marks. Find the **three** possible values of c .

Solution:

Since 4 is the mode, then $c < 9$

The total number of students is $8 + 7 + c + 9 + 1 = c + 25$

Since 3 is the median, then $\frac{c+25}{2} > 8+7 \Rightarrow c+25 > 30 \Rightarrow c > 5$

Thus, the three possible values of c are 6, 7, 8

6. [Maximum mark: 8]

Let $f(x) = x \ln\left(\frac{e}{2x}\right)$. Point A is on the curve of f where $x=1$. Point B is also on the curve of f . The tangent to the curve of f at A is perpendicular to the tangent at B. Find the coordinates of B.

Solution:

$$f'(x) = \ln\left(\frac{e}{2x}\right) + x \left(\frac{2x}{e} \cdot \frac{d}{dx} \left(\frac{e}{2x} \right) \right) = \ln\left(\frac{e}{2x}\right) + x \left(\frac{2x}{e} \left(-\frac{e}{2x^2} \right) \right) = \ln\left(\frac{e}{2x}\right) - 1 = \ln e - \ln(2x) - 1$$

$$\text{Thus, } f'(x) = -\ln(2x)$$

$$\text{Gradient of tangent at A: } f'(1) = -\ln(2)$$

$$\text{Hence, gradient of tangent at B is } \frac{1}{\ln(2)}$$

Find value of x such that the gradient of tangent to f is equal to $\frac{1}{\ln(2)}$.

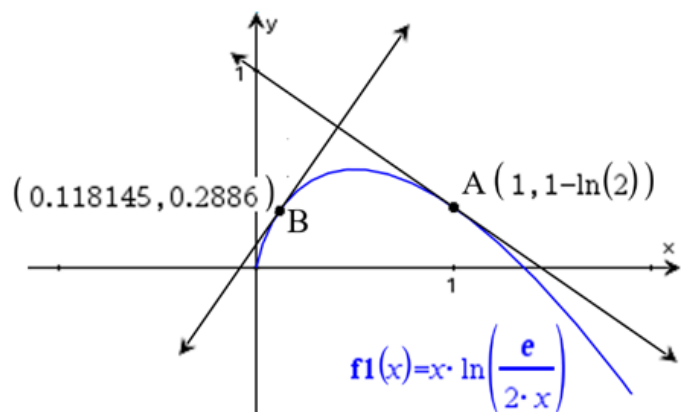
$$\text{Solve } f'(x) = -\ln(2x) = \frac{1}{\ln(2)} \quad \text{nSolve}\left(-\ln(2 \cdot x) = \frac{1}{\ln(2)}, x\right) \quad 0.118145044172$$

$$f(0.118145044172...) \approx 0.288592313505...$$

Thus, coordinates of B are (0.118, 0.289)

note: exact answer is $B\left(\frac{1}{2e^{\frac{1}{\ln 2}}}, \frac{1 + \ln 2}{2e^{\frac{1}{\ln 2}} \ln 2}\right)$, but this requires an unnecessary amount of work

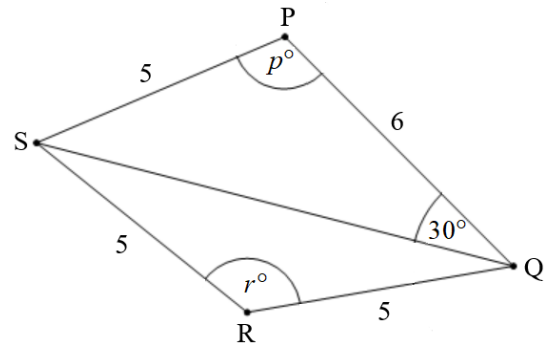
Sketch illustrates that the tangents at A and B appear to be perpendicular



7. [Maximum mark: 15]

The diagram shows the quadrilateral PQRS.
 Angle QPS and angle QRS are obtuse.

$PQ = 6 \text{ cm}$, $QR = 5 \text{ cm}$, $RS = 5 \text{ cm}$, $PS = 5 \text{ cm}$, $\widehat{PQS} = 30^\circ$,
 $\widehat{QPS} = p^\circ$, $\widehat{QRS} = r^\circ$



- (a) Use the sine rule to show that $QS = 10 \sin p$. [1]
- (b) Use the cosine rule in triangle PQS to find another expression for QS. [3]
- (c) (i) Hence, find p , giving your answer to two decimal places.
 (ii) Find QS. [6]
- (d) (i) Find r .
 (ii) Hence, or otherwise, find the area of triangle QRS. [5]

Solution:

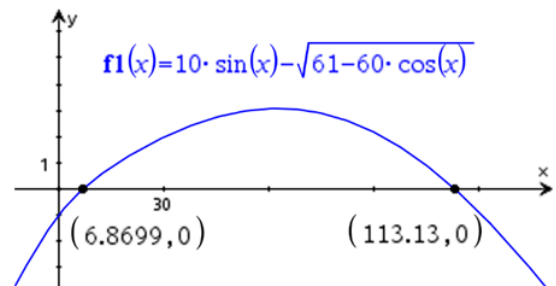
(a) $\frac{\sin 30^\circ}{5} = \frac{\sin p}{QS} \Rightarrow QS = \frac{5}{\frac{1}{2}} \sin p \Rightarrow QS = 10 \sin p \quad \text{QED}$

(b) $QS^2 = 5^2 + 6^2 - 2(5)(6)\cos p \Rightarrow QS = \sqrt{61 - 60 \cos p}$

(c) (i) solve the equation: $10 \sin p = \sqrt{61 - 60 \cos p}$
 clearly, $90 < p < 180$; solve with graph or GDC solver

```
nSolve(10 * sin(p) = sqrt(61 - 60 * cos(p)), p) | p > 6.87
113.130102354
```

thus, $p \approx 113.13$



(c) (ii) $QS = \sqrt{61 - 60 \cos(113.130102354^\circ)} \approx 9.19615... \Rightarrow QS \approx 9.20 \text{ cm}$

(d) (i) using cosine rule in triangle QRS

$$(9.19615...)^2 = 5^2 + 5^2 - 2(5)(5)\cos r \Rightarrow r = \cos^{-1}\left(\frac{50 - (9.19615...)^2}{50}\right) \approx 133.7398... \Rightarrow r \approx 134$$

(ii) area of triangle QRS = $\frac{1}{2}(5)(5)\sin(133.7398...^\circ) \approx 9.0310889... \approx 9.03 \text{ cm}^2$

8. [Maximum mark: 14]

A commercial plantation grows pineapples that are classified as small, medium or large. The masses of the pineapples harvested in the year 2021 were normally distributed with a mean of 900 grams.

A pineapple is small if its mass is less than 750 grams. For the year 2021, sixteen percent of the pineapples are classified as small.

- (a) Find the standard deviation of the masses of the pineapples. [4]

The following table shows the percentages of small, medium and large pineapples grown on the plantation in 2021.

small	medium	large
16%	63%	21%

- (b) Given that a pineapple is classified as medium if its mass is greater than 750 grams and less than m grams, find the value of m . [2]

The plantation sends a shipment containing all of its medium and large pineapples harvested in 2021 to a food distribution company.

- (c) A pineapple is randomly selected from this shipment. Find the probability that the pineapple is classified as medium. [3]
- (d) The food distribution company sells all the pineapples in the shipment. It sells each of the medium pineapples for \$3.30 and each of the large pineapples for \$4.10. The food distribution company paid \$900 for the shipment and makes a profit of \$500 after selling all the pineapples in the shipment. Find the total number of pineapples in the shipment. [5]

Solution:

(a) $X \sim N(900, \sigma)$ and $P(X < 750) = 0.16 \Rightarrow Z \approx -0.994458\dots$

$$Z = \frac{x - \mu}{\sigma} \Rightarrow -0.994458\dots = \frac{750 - 900}{\sigma} \Rightarrow \sigma = \frac{-150}{-0.99446\dots} \approx 150.836\dots \Rightarrow \sigma \approx 151 \text{ grams}$$

(b) $P(\text{small} + \text{medium} < m) = 0.16 + 0.63 = 0.79 \Rightarrow Z \approx 0.806421\dots$

$$0.80642\dots = \frac{x - 900}{150.836\dots} \Rightarrow x = (0.80642\dots)(150.836\dots) + 900 \approx 1021.64\dots \Rightarrow m \approx 1020 \text{ grams}$$

OR, in one step on GDC:

Inverse Normal	
Area:	0.79
μ :	900
σ :	150.8359

$$\text{invNorm}(0.79, 900, 150.8359) = 1021.63727513$$

- (c) Find probability of selecting a medium pineapple given it's in the shipment of medium & large

$$P(\text{medium} | \text{in shipment}) = \frac{P(\text{medium} \cap \text{in shipment})}{P(\text{in shipment})} = \frac{0.63}{0.63 + 0.21} = \frac{0.63}{0.84} = 0.75 \text{ or } \frac{3}{4}$$

- (d) $P(\text{large}) = 1 - 0.75 = 0.25$ total sales - 900 = 500 \rightarrow total sales = 1400

Let x be the number of pineapples in the shipment:

$$3.30(0.75x) + 4.10(0.25x) = 1400 \Rightarrow 3.5x = 1400 \Rightarrow x = \frac{1400}{3.5} = 400 \text{ pineapples in shipment}$$

9. [Maximum mark: 14]

- (a) Given that $h(x) = \frac{ax-1}{bx-b}$, find the equation of the vertical asymptote and the equation for the horizontal asymptote for the graph of h . [2]

The vertical and horizontal asymptotes for the graph of h intersect at the point F.

- (b) Write down the coordinates of F. [2]

- (c) The point $G(x, y)$ lies on the graph of h . Show that $FG = \sqrt{(x-1)^2 + \left(\frac{a-1}{bx-b}\right)^2}$. [4]

- (d) Hence, find the coordinates of the points on the graph of $y = \frac{4x-1}{2x-2}$ that are closest to the point $(1, 2)$. [6]

Solution:

- (a) vertical asymptote: $x = 1$, horizontal asymptote: $y = \frac{a}{b}$

- (b) $F\left(1, \frac{a}{b}\right)$

- (c) applying distance formula for distance between $F\left(1, \frac{a}{b}\right)$ and $G(x, y)$

$$FG = \sqrt{(x-1)^2 + \left(y - \frac{a}{b}\right)^2} \quad \text{substitute } \frac{ax-1}{bx-b} \text{ for } y$$

$$\begin{aligned} FG &= \sqrt{(x-1)^2 + \left(\frac{ax-1}{bx-b} - \frac{a}{b}\right)^2} = \sqrt{(x-1)^2 + \left(\frac{ax-1}{b(x-1)} - \frac{a(x-1)}{b(x-1)}\right)^2} \\ &= \sqrt{(x-1)^2 + \left(\frac{ax-1-ax+a}{bx-b}\right)^2} = \sqrt{(x-1)^2 + \left(\frac{a-1}{bx-b}\right)^2} \quad \text{QED} \end{aligned}$$

- (d) $y = \frac{4x-1}{2x-2} \Rightarrow a = 4, b = 2 \Rightarrow F$ is at $(1, 2)$; thus, the distance from $(1, 2)$ to any

point $G(x, y)$ on the graph of h is given by $FG = \sqrt{(x-1)^2 + \left(\frac{3}{2x-2}\right)^2}$

graph of expression for FG shows minimum

occurs at $x \approx -0.22474\dots$ and $x \approx 2.22474\dots$

$$y(-0.22474\dots) \approx 0.7753\dots, \quad y(2.22474\dots) \approx 3.2247\dots$$

Thus, coordinates of the 2 points on $y = \frac{4x-1}{2x-2}$

that are closest to $(1, 2)$ are

$$(-0.225, 0.775) \text{ and } (2.22, 3.22)$$

